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On high-frequency oscillatory viscous flows

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The equations governing high-frequency oscillatory viscous flows are investigated through the separation of the steady and the unsteady parts. All Reynolds number ranges are studied and the orders of magnitude of the steady streaming produced by the Reynolds stresses are established.

The oscillating circular cylinder at low Reynolds numbers is studied through the method of inner and outer expansions. Steady recirculating cells exist near the cylinder. The results compare very well with experiments. Analytic expressions for the streamfunction and the drag coefficient are obtained.

The oscillating flow towards an infinite plate is investigated in detail. The steady streaming is caused by the steady component of the Reynolds stress. The pressure gradient always causes reverse flow near the solid boundary.

1. Introduction

The striking result of the interaction of an oscillatory viscous flow with a solid boundary is the induction of a time-independent streaming motion. This induced steady streaming is generated by the non-linear Reynolds stresses in the unsteady boundary layer.

The first experiment and theory of such streaming motion can probably be dated back to Faraday (1831) and Rayleigh (1883), who studied the steady motions caused by vibrating plates. Since the translatory oscillations of a fluid past an object are an approximation to sound waves whose wavelength is very large compared with the dimensions of the object, this topic is very important in the field of acoustics. Many papers have been written in this century on the steady streaming (acoustic streaming) caused by an oscillatory viscous flow. References to early literature were cited by Westervelt (1953) and Nyborg (1953).

The more important theoretical treatments of acoustic streaming are probably those due to Schlichting (1932) and Holtsmark *et al.* (1954). Recently the problem was again discussed by Stuart (1966). From the usual concepts of boundary-layer theory Stuart showed that, for high Reynolds numbers, in addition to the unsteady boundary layer there exists a second boundary layer in which the steady streaming decays to zero.

However, many points remain to be answered. In this paper we shall re-examine the equations governing oscillatory viscous flows. The assumptions we

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shall make are that the reduced frequency of oscillation is large and that the fluid is incompressible.

We shall fix our co-ordinates on the solid body, and regard the fluid as oscillating. The problem is then completely governed by two parameters: a Reynolds number R characterizing viscosity and a Strouhal number S characterizing frequency. These are defined as follows:

$$R = U_\infty l/\nu, \quad S = l\omega/U_\infty.$$

Here U_∞ is the velocity amplitude of the oscillating fluid at infinity, l is a characteristic length of the body (the radius of a cylinder if S is to be interpreted as the ratio of the oscillation amplitude to the diameter) ω is the frequency of oscillation and ν is kinematic viscosity.

2. The governing equations

It will be convenient to eliminate pressure and write the unsteady Navier–Stokes equations in terms of vorticity:

$$\frac{\partial \boldsymbol{\zeta}'}{\partial t} - \nabla \times \mathbf{q}' \times \boldsymbol{\zeta}' = -\nu \nabla \times \nabla \times \boldsymbol{\zeta}', \tag{2.1}$$

$$\nabla . \mathbf{q}' = 0. \tag{2.2}$$

Here the vorticity is defined as $\boldsymbol{\zeta}' = (\nabla \times \mathbf{q}')$. Before a comparison of magnitudes of the terms can be made, we must separate the steady part of each variable from its unsteady part. (This can be done by differentiating and then integrating with respect to time.) Denoting the steady part by a bar and the unsteady part by a tilde, and assuming each variable is the sum of the above two parts, we have

$$\frac{\partial \tilde{\boldsymbol{\zeta}}'}{\partial t} - \nabla \times \bar{\mathbf{q}}' \times \tilde{\boldsymbol{\zeta}}' - \nabla \times \tilde{\mathbf{q}}' \times \bar{\boldsymbol{\zeta}}' - (\nabla \times \tilde{\mathbf{q}}' \times \tilde{\boldsymbol{\zeta}}')_u = -\nu \nabla \times \nabla \times \tilde{\boldsymbol{\zeta}}', \tag{2.3}$$

$$\nabla . \tilde{\mathbf{q}}' = 0, \tag{2.4}$$

$$(\nabla \times \tilde{\mathbf{q}}' \times \tilde{\boldsymbol{\zeta}}')_s + \nabla \times \bar{\mathbf{q}}' \times \bar{\boldsymbol{\zeta}}' = \nu \nabla \times \nabla \times \bar{\boldsymbol{\zeta}}', \tag{2.5}$$

$$\nabla . \bar{\mathbf{q}}' = 0. \tag{2.6}$$

Here $(\ )_u$ and $(\ )_s$ denote the unsteady part and the steady part of the product, respectively. The interactions between the unsteady flow and the steady flow can be seen at once from (2.3) and (2.5). The term $(\nabla \times \tilde{\mathbf{q}}' \times \tilde{\boldsymbol{\zeta}}')_s$ is associated with the steady part of the Reynolds stress. In non-periodic unsteady flows, this term is zero. For periodic unsteady flow this term becomes the forcing function for the steady streaming motion.

The boundary conditions are that the velocities approach a prescribed oscillatory flow at infinity and that the velocities are zero on the solid surface.

Since we have purely oscillatory boundary conditions, we do not know *a priori* what order of magnitude is the steady streaming velocity. Let us denote an unknown γ as the ratio between order of magnitude of the steady velocity to that of the unsteady velocity. Then we normalize the unsteady velocity by U_∞, the steady velocity by γU_∞, the time by $1/\omega$, the length by l and drop the primes. We obtain

$$\frac{\partial \tilde{\boldsymbol{\zeta}}}{\partial t}-\frac{\gamma}{S}(\nabla\times\bar{\mathbf{q}}\times\tilde{\boldsymbol{\zeta}}+\nabla\times\tilde{\mathbf{q}}\times\bar{\boldsymbol{\zeta}})-\frac{1}{S}(\nabla\times\tilde{\mathbf{q}}\times\tilde{\boldsymbol{\zeta}})_u=-\frac{1}{RS}(\nabla\times\nabla\times\tilde{\boldsymbol{\zeta}}), \quad (2.7)$$

$$\nabla.\tilde{\mathbf{q}}=0, \quad (2.8)$$

$$(\nabla\times\tilde{\mathbf{q}}\times\tilde{\boldsymbol{\zeta}})_s+\gamma^2(\nabla\times\bar{\mathbf{q}}\times\bar{\boldsymbol{\zeta}})=(\gamma/R)(\nabla\times\nabla\times\bar{\boldsymbol{\zeta}}), \quad (2.9)$$

$$\nabla.\bar{\mathbf{q}}=0, \quad (2.10)$$

and the unprimed variables are of order unity.

In (2.7), if $RS \leqslant O(1)$, the unsteady vorticity $\tilde{\boldsymbol{\zeta}}$ would not be confined in a boundary layer but would be spread all over the flow field. Then from (2.9) the forcing term $(\nabla\times\tilde{\mathbf{q}}\times\tilde{\boldsymbol{\zeta}})_s$ is also important in the entire region. To balance this term, the ratio γ in the last term of (2.9) should be of order $O(R)$. In this case no boundary layer exists. For large values of S the equations of motion reduce to

$$\frac{\partial \tilde{\boldsymbol{\zeta}}}{\partial t}+\alpha(\nabla\times\nabla\times\tilde{\boldsymbol{\zeta}})=\epsilon(\nabla\times\tilde{\mathbf{q}}\times\tilde{\boldsymbol{\zeta}})_u+\frac{\epsilon^2}{\alpha}(\nabla\times\bar{\mathbf{q}}\times\tilde{\boldsymbol{\zeta}}+\nabla\times\tilde{\mathbf{q}}\times\bar{\boldsymbol{\zeta}}), \quad (2.11)$$

$$(\nabla\times\nabla\times\bar{\boldsymbol{\zeta}})=(\nabla\times\tilde{\mathbf{q}}\times\tilde{\boldsymbol{\zeta}})_s+\frac{\epsilon^2}{\alpha^2}(\nabla\times\bar{\mathbf{q}}\times\bar{\boldsymbol{\zeta}}), \quad (2.12)$$

where $R \ll 1$, $\epsilon = 1/S \ll 1$ and $\alpha = 1/RS$ is a constant of order unity. The zeroth-order perturbation of (2.11) and (2.12) is essentially the theoretical formulation adopted by Rayleigh (1883), Holtsmark *et al.* (1954) and Lane (1955). For cylinders and spheres, the solutions involve Hankel functions which must be integrated numerically. One must, however, consider the interaction terms which affect the $O(\epsilon^2)$ unsteady vorticity equation in (2.11).

Let us consider the case when $(RS) > O(1)$. Equation (2.7) then shows an unsteady boundary layer of $O(RS)^{-\frac{1}{2}}$ exists, which implies that the unsteady vorticity decays exponentially outside this boundary layer. The forcing term of the steady flow $(\nabla\times\tilde{\mathbf{q}}\times\tilde{\boldsymbol{\zeta}})_s$ is also negligible outside the distance $(RS)^{-\frac{1}{2}}$. In order to balance this force inside this distance, γ must be equal to $(1/S)$. Equation (2.9) then becomes

$$(\nabla\times\tilde{\mathbf{q}}\times\tilde{\boldsymbol{\zeta}})_s+\frac{1}{S^2}(\nabla\times\bar{\mathbf{q}}\times\bar{\boldsymbol{\zeta}})=\frac{1}{RS}(\nabla\times\nabla\times\bar{\boldsymbol{\zeta}}). \quad (2.13)$$

Outside the distance $(RS)^{-\frac{1}{2}}$ equation (2.13) is replaced by

$$\frac{1}{S^2}(\nabla\times\bar{\mathbf{q}}\times\bar{\boldsymbol{\zeta}})=\frac{1}{RS}(\nabla\times\nabla\times\bar{\boldsymbol{\zeta}}). \quad (2.14)$$

The boundary condition on (2.14) is that the velocity $\bar{\mathbf{q}}$ on the body matches a non-zero steady velocity produced by the unsteady Reynolds stress. We can see from (2.14) that a 'second boundary layer' exists or not depending on the parameter $R/S = U_\infty^2/\omega\nu$. This parameter is called steady streaming Reynolds number R_s by Stuart (1966). If $R/S < O(1)$ the outer steady flow field is governed by the Stokes equation. If $R/S > O(1)$, a 'second' boundary layer exists. The vorticity $\bar{\boldsymbol{\zeta}}$ then decays exponentially outside the steady boundary layer of $O(R/S)^{-\frac{1}{2}}$. The steady velocity, if any, is potential outside this boundary layer. On the other hand, if $R/S = O(1)$ we must take into account the vorticity transport. Then the steady streaming is governed by the full Navier–Stokes equations.

The importance of curvature terms in (2.13) is of order $(RS)^{-\frac{1}{2}}$. Thus the curvature effects become important when $O(R) \leqslant O(S)$. This has been pointed out by Wang (1966) in reference to Schlichting's (1932) theory. Schlichting used the Stokes formulation for the outer steady streaming. If the curvature terms

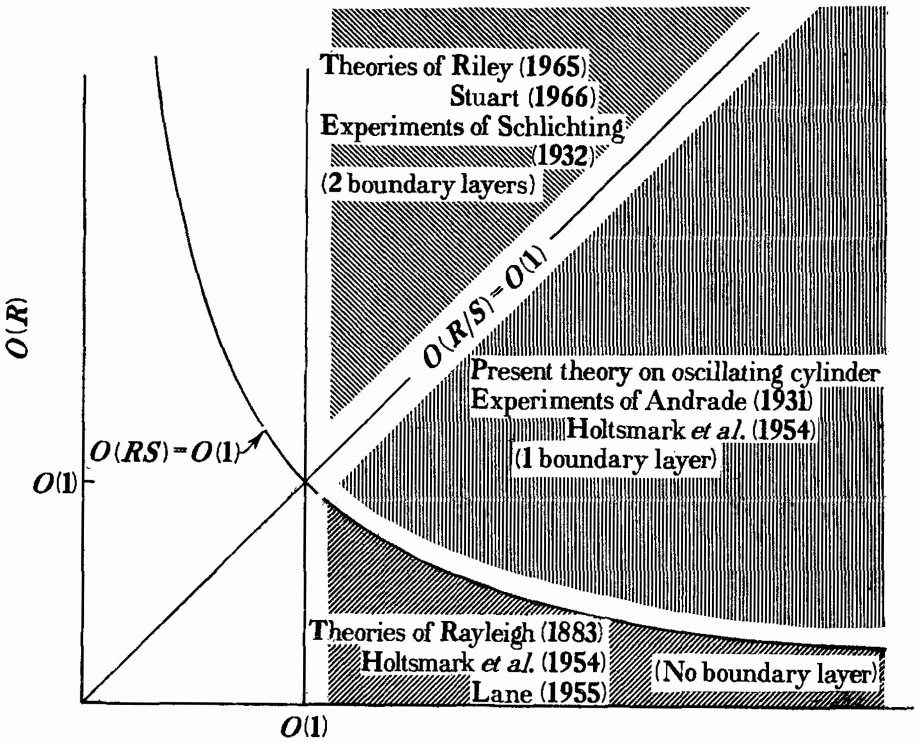

FIGURE 1. The regimes of validity of various theories and experiments.

were retained, Schlichting's theory should be valid for $R/S \ll 1$ or very small R_s compared to unity in Stuart's notation. Wang (1965) also studied the case $O(R) = O(S)$ for a sphere. However, the outer steady streaming was done incorrectly. This outer flow should be governed by the full Navier–Stokes equations.

Experiments in the range $RS \gg 1$ and $R/S \ll 1$ have been done by Andrade (1931) for a sphere ($S \sim 100$, $R \sim 5$), and by Holtsmark *et al.* (1954) for a cylinder (typical case: ($S \simeq 17{\cdot}25$, $R = 5{\cdot}87$)).

The case $RS \gg 1$, $R/S \gg 1$ was considered theoretically by Stuart (1966) and Riley (1965). Stuart used Fettis' (1955) method to calculate the steady outer boundary layer. He especially referred to Schlichting's (1932) experiments ($RS = 4250$, $R/S = 53{\cdot}7$). Riley used an 'inner and outer expansions' method. Both Stuart and Riley expanded the boundary-layer solutions from a steady stagnation point $\frac{1}{2}\pi$ from the unsteady stagnation point.

The regimes of validity of various theories and experiments are shown in figure 1.

One must also mention the related problem of the *infinite* oscillating disk by Rosenblat (1959) and Benney (1964). Since there are no natural length scales, theoretically this solution is valid for all Reynolds numbers. The governing equations are similar to that obtained by taking Reynolds number equal to unity. Thus from the above arguments the outer steady streaming should be governed by the Stokes equation. Physically, as no disk is infinite, the solution is only valid in a small region near the axis and the finite disk. Outside the region of the infinite disk solution, convection becomes important and the Stokes equation is inadequate to describe the flow at high Reynolds numbers. Therefore for a *finite* disk there may exist another layer where convection is important.

3. Oscillating circular cylinder at low Reynolds number and large Strouhal number

To illustrate, take the oscillating circular cylinder in an otherwise still fluid. We shall restrict ourselves to small Reynolds numbers such that $RS \gg 1$, $R/S \ll 1$, where some careful experiments have been done.

Curvature effects, represented through the Reynolds number, enter as an important factor. There are two direct consequences due to finite curvature. First, the slope of the original outer oscillating flow affects the next-order solution. Secondly, the induced outer flow due to displacement has a tangential component of the velocity, which in turn creates its own boundary layer. These will be illustrated in this example.

When the problem involves two parameters, it is advisable to expand in one while assuming some proportional relationship with the other. For our problem, we define

$$1/S \equiv \epsilon^2, \tag{3.1}$$

$$R/S = O(\epsilon^2), \tag{3.2}$$

or

$$R/S \equiv \epsilon^2/\alpha, \tag{3.3}$$

where ϵ is a small number and α is a constant of order unity. From previous arguments $\gamma = 1/S$ and the governing equations, in cylindrical polar co-ordinates fixed on the cylinder, become

$$\frac{\partial \zeta}{\partial t} - \alpha\epsilon^2\nabla^2\zeta = -\frac{\epsilon^4}{r}\left[\frac{\partial(\overline{\psi}, \zeta)}{\partial(r,\theta)} + \frac{\partial(\tilde{\psi}, \tilde{\zeta})}{\partial(r,\theta)}\right] - \frac{\epsilon^2}{r}\left[\frac{\partial(\tilde{\psi}, \tilde{\zeta})}{\partial(r,\theta)}\right]_u, \tag{3.4}$$

$$\alpha\epsilon^2\nabla^2\bar{\zeta} = \frac{\epsilon^4}{r}\frac{\partial(\overline{\psi}, \bar{\zeta})}{\partial(r,\theta)} + \frac{1}{r}\left[\frac{\partial(\tilde{\psi}, \tilde{\zeta})}{\partial(r,\theta)}\right]_s, \tag{3.5}$$

where ψ is the streamfunction and

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \tag{3.6}$$

Using the method of inner and outer expansions, we perturb the vorticity and the streamfunction as follows:

$$\zeta = z_0 + \epsilon z_1 + \epsilon^2 z_2 + \epsilon^3 z_3 + \epsilon^4 z_4 + \ldots, \tag{3.7}$$

$$\psi = \Psi_0 + \epsilon\Psi_1 + \epsilon^2\Psi_2 + \ldots. \tag{3.8}$$

Substituting into (3.4), we find the unsteady vorticity is identically zero until $O(\epsilon^4)$, where the interactions come into play:

$$\tilde{z}_0 = \tilde{z}_1 = \tilde{z}_2 = \tilde{z}_3 = 0, \tag{3.9}$$

$$\frac{\partial \tilde{z}_4}{\partial t} = -\frac{1}{r}\frac{\partial(\tilde{\Psi}_4, \bar{z}_0)}{\partial(r,\theta)}. \tag{3.10}$$

From (3.6) the steady vorticity is governed by the Stokes equation for the first few orders until $O(\epsilon^2)$:

$$\nabla^2 \bar{z}_0 = \nabla^2 \bar{z}_1 = 0, \tag{3.11}$$

$$\alpha \nabla^2 \bar{z}_2 = \frac{1}{r}\frac{\partial(\bar{\Psi}_0, \bar{z}_0)}{\partial(r,\theta)} + \frac{1}{r}\left[\frac{\partial(\tilde{\Psi}_0, \tilde{z}_4)}{\partial(r,\theta)}\right]_s. \tag{3.12}$$

The equations for the inner flow field are obtained by stretching in the radial direction as follows:

$$r = 1 + \epsilon \alpha^{\frac{1}{2}} \eta, \tag{3.13}$$

$$\psi = \epsilon(\psi_0 + \epsilon \psi_1 + \epsilon^2 \psi_2 + \ldots). \tag{3.14}$$

The equations are

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial \eta^2}\right)\frac{\partial^2}{\partial \eta^2}\tilde{\psi}_0 = 0, \tag{3.15}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial \eta^2}\right)\frac{\partial^2}{\partial \eta^2}\tilde{\psi}_1 = \alpha^{\frac{1}{2}}\left(2\frac{\partial^3}{\partial \eta^3} - \frac{\partial^2}{\partial t\,\partial \eta}\right)\tilde{\psi}_0, \tag{3.16}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial \eta^2}\right)\frac{\partial^2}{\partial \eta^2}\tilde{\psi}_2 = \alpha^{\frac{1}{2}}\left(2\frac{\partial^3}{\partial \eta^3} - \frac{\partial^2}{\partial t\,\partial \eta}\right)\tilde{\psi}_1 - \alpha\left(2\eta\frac{\partial^3}{\partial \eta^3} + \frac{\partial^2}{\partial \eta^2} - \eta\frac{\partial^2}{\partial t\,\partial \eta}\right)\tilde{\psi}_0 - \frac{1}{\alpha^{\frac{1}{2}}}\left[\frac{\partial(\tilde{\psi}_0, \tilde{\psi}_{0\eta\eta})}{\partial(\eta,\theta)}\right]_u, \tag{3.17}$$

$$\frac{\partial^4}{\partial \eta^4}\bar{\psi}_0 = \frac{1}{\alpha^{\frac{1}{2}}}\left[\frac{\partial(\tilde{\psi}_0, \tilde{\psi}_{0\eta\eta})}{\partial(r,\theta)}\right]_s, \tag{3.18}$$

etc. The boundary conditions are that the velocities are zero on the cylinder described by $r = 1$ and that the velocities approach a uniform oscillation at infinity:

$$\tilde{\Psi}_0\big|_{r=\infty} = r \sin\theta\, e^{it}, \tag{3.19}$$

$$\Psi_n\big|_{r=\infty} = 0 \quad (n \neq 0), \tag{3.20}$$

where it is understood that only the real part has any physical significance. We shall first solve for the unsteady flow. From (3.9), (3.19) and (3.15) we have

$$\tilde{\Psi}_0 = \sin\theta\left(r - \frac{1}{r}\right)e^{it}, \tag{3.21}$$

$$\tilde{\psi}_0 = 2\,\alpha^{\frac{1}{2}} \sin\theta\left[\eta + \frac{1-i}{\sqrt{2}}(E-1)\right]e^{it}, \tag{3.22}$$

where E stands for $e^{-\eta(1+i)/\sqrt{2}}$. From the matching condition

$$\lim_{r\to 1}(\tilde{\Psi}_0 + \epsilon\tilde{\Psi}_1 + \epsilon^2\tilde{\Psi}_2 + \ldots) \quad \text{matches} \quad \lim_{\eta\to\infty}\epsilon(\tilde{\psi}_0 + \epsilon\tilde{\psi}_1 + \ldots), \tag{3.23}$$

we obtain

$$\tilde{\Psi}_1\big|_{r=1} = (i-1)(2\alpha)^{\frac{1}{2}} \sin\theta\, e^{it}. \tag{3.24}$$

The solution to (3.9), using the boundary conditions of (3.20) and (3.24), is a doublet:

$$\tilde{\Psi}_1 = -(2\alpha)^{\frac{1}{2}}(1-i)\frac{\sin\theta}{r}e^{it}. \tag{3.25}$$

This doublet is due to the oscillating displacement thickness of the unsteady boundary layer. To an observer at infinity, the cylinder seems to be pulsating with the same frequency as the oscillating flow. Notice that (3.25) introduces a non-zero tangential velocity component which is to be matched with the inner solution $\tilde{\psi}_1$. These curvature effects are very important.

The solution to (3.16) is

$$\tilde{\psi}_1 = \alpha \sin\theta \left\{ -\eta^2 + \sqrt{2}(1-i)\,\eta + i - iE - \frac{(1-i)}{\sqrt{2}}\,\eta E \right\} e^{it}. \tag{3.26}$$

The second-order induced outer flow is again an oscillating doublet which will influence the second-order boundary layer

$$\tilde{\Psi}_2 = i\alpha\,\frac{\sin\theta}{r}\,e^{it}. \tag{3.27}$$

To calculate the non-linear Reynolds stress terms in (3.17) we must do the multiplication in the real physical domain, then return to complex variables:

$$\left[\frac{\partial(\tilde{\psi}_0, \tilde{\psi}_{0\eta\eta})}{\partial(\eta,\theta)}\right]_u = i\alpha \sin 2\theta\eta E\, e^{2it}, \tag{3.28}$$

$$\left[\frac{\partial(\tilde{\psi}_0, \tilde{\psi}_{0\eta\eta})}{\partial(\eta,\theta)}\right]_s = \alpha \sin 2\theta\,[\sqrt{2}\,E - \sqrt{2}\,e^{-\sqrt{2}\eta} + i\eta E]. \tag{3.29}$$

From (3.17) the second-order inner solution is found to be

$$\tilde{\psi}_2 = \alpha^{\frac{3}{2}} \sin\theta \left\{ \eta^3 + \sqrt{2}(i-1)\,\eta^2 - i\eta + \frac{1+i}{4\sqrt{2}}\,[E-1] + \tfrac{5}{4} i\eta E + \frac{3}{4\sqrt{2}}\,(1-i)\,\eta^2 E \right\} e^{it}$$
$$+ \alpha^{\frac{1}{2}} \sin 2\theta \left\{ \frac{1+i}{2}\,[E^{\sqrt{2}} - 1] + i\eta E \right\} e^{2it}. \tag{3.30}$$

We see that the non-linear transport terms enter in the second-order unsteady equations. The displacement of (3.30) not only introduces a doublet oscillating with the basic frequency, but also a quadrupole oscillating with twice the basic frequency,

$$\tilde{\Psi}_3 = -\frac{1+i}{4\sqrt{2}}\,\alpha^{\frac{3}{2}}\,\frac{\sin\theta}{r}\,e^{it} - \frac{1+i}{2}\,\alpha^{\frac{1}{2}}\,\frac{\sin 2\theta}{r^2}\,e^{2it}. \tag{3.31}$$

Notice that to this order we can solve independently for the unsteady flow field. The interaction of the steady flow field has not entered yet. To find the steady flow, we use a similar procedure outlined above and, from (3.18) and (3.29),

$$\overline{\psi}_0 = \alpha^{\frac{1}{2}} \sin 2\theta \left[-\tfrac{3}{2}\eta + \frac{13}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\,e^{-\sqrt{(2\eta)}} - \sqrt{2}(3+2i)\,E - i\eta E \right]. \tag{3.32}$$

Equation (3.32) shows that there exists a finite tangential velocity of $(-\frac{3}{2}\sin 2\theta)$ outside the boundary layer. From (3.11) the outer steady flow is governed by the Stokes equations. The solution after matching is

$$\overline{\Psi}_0 = \tfrac{3}{4} \sin 2\theta \left(\frac{1}{r^2} - 1\right). \tag{3.33}$$

Since the forcing function decays exponentially outside the boundary layer, $\bar{\Psi}_0$ represents the fluid being 'dragged along' by the velocity just outside the boundary layer. Of different nature is the flow $\bar{\Psi}_1$ which is induced by the displacement of the steady boundary layer

$$\bar{\Psi}_1 = \alpha^{\frac{1}{2}} \frac{13}{2\sqrt{2}} \frac{\sin 2\theta}{r^2}. \tag{3.34}$$

$\bar{\Psi}_1$ of course produces its own boundary layer. The method is systematic and we shall not pursue it further.

Uniformly valid solutions of the entire region for the unsteady flow and the steady flow (of order γ or order $1/S$ smaller) are constructed:

$$\begin{aligned}\tilde{\psi} = {} & \sin\theta\left(r-\frac{1}{r}\right)e^{it} + (RS)^{-\frac{1}{2}}\sqrt{2}(1-i)\sin\theta\, E\, e^{it} \\ & -(RS)^{-\frac{1}{2}}\sqrt{2}(1-i)\frac{\sin\theta}{r}e^{it} - (RS)^{-1}\sin\theta\left[iE+\frac{1-i}{\sqrt{2}}\eta E\right]e^{it} \\ & +(RS)^{-1}i\frac{\sin\theta}{r}e^{it} + (RS)^{-\frac{3}{2}}\sin\theta\left\{\frac{1+i}{4\sqrt{2}}[E-1]+\frac{5i}{4}\eta E+\frac{3}{4\sqrt{2}}(1-i)\eta^2 E\right\}e^{it} \\ & +R^{-\frac{1}{2}}S^{-\frac{3}{2}}\sin 2\theta\{\tfrac{1}{2}(1+i)[E^{\sqrt{2}}-1]+i\eta E\}e^{2it}+O(S^{-2}),\end{aligned} \tag{3.35}$$

$$\begin{aligned}\bar{\psi} = {} & S^{-1}\tfrac{3}{4}\sin 2\theta\left(\frac{1}{r^2}-1\right)+R^{-\frac{1}{2}}S^{-\frac{3}{2}} \\ & \times\sin 2\theta\left[\frac{13}{2\sqrt{2}}-\frac{1}{2\sqrt{2}}e^{-\sqrt{2}\eta}-\sqrt{2}(3+2i)E\right]+O(S^{-2}).\end{aligned} \tag{3.36}$$

The unsteady part of the streamfunction, (3.35), is dependent on both R and S. The first term on the right-hand side is the prescribed oscillatory flow. The other terms are due to the viscous interactions with the body. Multiples of the basic harmonic are present in the higher orders. Outside the boundary layer there exist induced oscillatory flows, represented by doublets and quadrupoles. The steady part, (3.36), is plotted in figure 2 for $R = 1$ and $S = 100$. Cells of recirculation exist symmetrically in each quadrant. This is consistent with experimental observations. Taking (3.36) to zero, we find that to order S^{-2} the position of the zeroth streamline depends only on the parameter RS. In figure 3 the thickness of the recirculating flow Δ is plotted against RS. The result is fairly good compared with direct observations (here normalized and replotted) by Holtsmark *et al.* (1954). Also shown in figure 3 is the theory by Holtsmark *et al.* obtained by a semi-numerical integration of Hankel functions. One must remark that, if solutions to (2.11) and (2.12) were expanded correctly, Holtsmark's work *should include* the present case.

The drag experienced by a circular cylinder in an oscillating stream can be found by integrating the pressure and the shear. The drag coefficient due to pressure is

$$\begin{aligned}C_p &= \frac{D_p}{(\frac{1}{2}\rho U_\infty^2)(2l\,\text{width})} = 2S\int_0^\pi p|_{\eta=0}\cos\theta\,d\theta \\ &= S2\pi i\,e^{it}+\left(\frac{S}{R}\right)^{\frac{1}{2}}\sqrt{2}\pi(1+i)\,e^{it}+\frac{\pi}{R}e^{it}+\frac{\pi}{R(RS)^{\frac{1}{2}}}\frac{i-1}{4\sqrt{2}}e^{it}+O(S^{-1}).\end{aligned} \tag{3.37}$$

The drag coefficient due to shear is:

$$C_s = \frac{D_s}{(\frac{1}{2}\rho U_\infty^2)(2l\,\text{width})} = 2\left(\frac{S}{R}\right)^{\frac{1}{2}} \int_0^\pi \left.\frac{\partial^2 \psi}{\partial r^2}\right|_{r=1} \sin\theta\, d\theta$$

$$= \left(\frac{S}{R}\right)^{\frac{1}{2}} \sqrt{2}\,\pi(1+i)\,e^{it} + \frac{\pi}{R}e^{it} + \frac{\pi}{R(RS)^{\frac{1}{2}}}\frac{i-1}{4\sqrt{2}}e^{it} + O(S^{-1}). \tag{3.38}$$

The first term in the right-hand side of (3.37) represents the drag due to the inviscid, potential flow $\tilde{\Psi}_0$. In the case of an oscillating cylinder in an otherwise still fluid, this term becomes half the present value and can be identified as the drag caused by the 'virtual mass' of the cylinder. The other terms in (3.37) and (3.38) are due to viscous interactions with curvature. Notice that in our problem the pressure drag dominates.

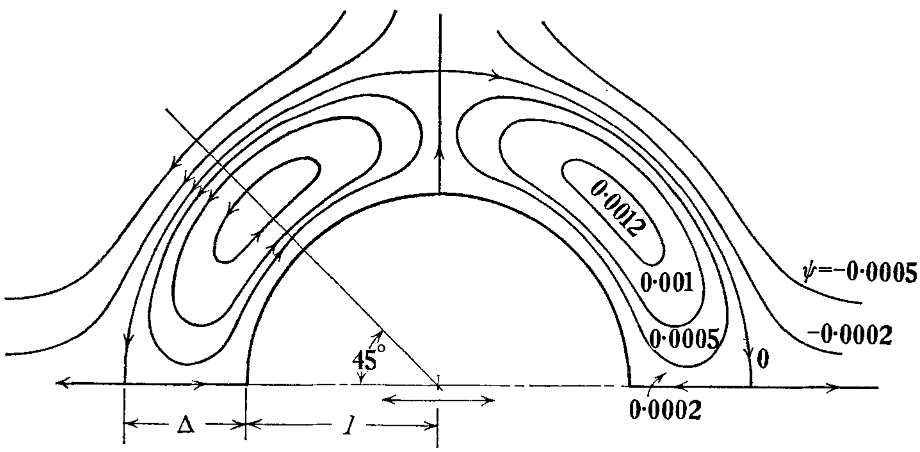

FIGURE 2. The steady streaming caused by an oscillating circular cylinder, $R = 1$, $S = 100$.

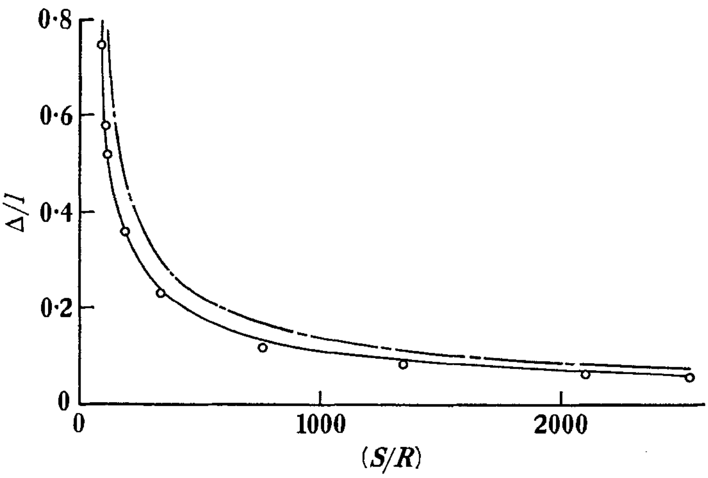

FIGURE 3. The thickness of recirculating cells versus the parameter (*SR*). ○, experiment (Holtsmark *et al.* 1954); — - — seminumerical theory (Holtsmark *et al.* 1954); ——, present theory.

The total drag experienced by an oscillating circular cylinder in an otherwise still fluid is then

$$C = C_p' + C_s' = S\pi i\, e^{it} + \left(\frac{S}{R}\right)^{\frac{1}{2}} 2\sqrt{2}\,\pi(1+i)\, e^{it} + \frac{2\pi}{R} e^{it} + \frac{\pi}{R(RS)^{\frac{1}{2}}} \frac{i-1}{2\sqrt{2}} e^{it} + O(S^{-1}). \tag{3.39}$$

It is interesting to note that the steady streaming, due to symmetry, does not contribute to drag. The energy for such streaming is embedded in the interaction terms in the $O(\epsilon^4)$ equations.

We must emphasize that the drag due to pressure, excluding that due to virtual mass, constitutes an important part of the total drag. This pressure drag can only be found from the induced outer flow obtained by asymptotic matching.

Riley (1966) also considered the oscillation of a sphere in the range $RS \gg 1$, $R/S \ll 1$. Following Schlichting (1932), he neglected the curvature terms which are very important in this case. Therefore, like Schlichting's work, the inner solution is valid for $R/S > O(1)$ but not for $R/S \leqslant O(1)$. This point is also discussed by Wang (1966). The zeroth-order outer steady flow, however, is not affected by curvature.

4. Oscillating flow at a stagnation point

As another example, we shall take the high-frequency oscillatory flow towards an infinite plane. The purpose of this example is threefold. First, since oscillatory flow parallel to a plane does not produce steady streaming, we believe that oscillatory stagnation flow towards a plane is more probable as a basic mechanism for steady streaming. Secondly, the geometry is very much simplified, and the solution can be found to higher orders, where we can investigate the interactions between the steady flow and the unsteady flow more closely. Thirdly, some difficulties associated with the infinite geometry are illustrated.

As in the case of the infinite oscillating disk, there are no natural length scales. Theoretically the solution is valid for all Reynolds numbers, mathematically the equations are similar to that obtained by taking Reynolds number unity and physically the solution is valid only in a small region near the stagnation point.

In two-dimensional oscillating stagnation-point flow the velocities at infinity are prescribed as $\tilde{U} = ax\cos\omega t$, $\tilde{V} = -ay\cos\omega t$. The flow is bounded by the plane $y = 0$. The explicit effects of curvature (and thus the Reynolds number) do not enter. If we normalize the velocities by $(a\nu)^{\frac{1}{2}}$, time by $1/\omega$, and lengths by $(\nu/a)^{\frac{1}{2}}$, the governing equations become

$$\frac{\partial\tilde{\zeta}}{\partial t} - \epsilon^2\nabla^2\tilde{\zeta} = -\epsilon^4\left[\frac{\partial(\bar{\psi},\tilde{\zeta})}{\partial(y,x)} + \frac{\partial(\tilde{\psi},\bar{\zeta})}{\partial(y,x)}\right] - \epsilon^2\left[\frac{\partial(\tilde{\psi},\tilde{\zeta})}{\partial(y,x)}\right]_u, \tag{4.1}$$

$$\epsilon^2\nabla^2\bar{\zeta} = \epsilon^4\frac{\partial(\bar{\psi},\bar{\zeta})}{\partial(y,x)} + \left[\frac{\partial(\tilde{\psi},\tilde{\zeta})}{\partial(y,x)}\right]_s, \tag{4.2}$$

which are similar to (3.4) and (3.5) if we take $\alpha = 1$. We have defined for convenience $\epsilon^2 = 1/S = a/\omega$. The boundary conditions are that the velocities be zero on $y = 0$ and that the velocities approach (and be no more singular than)

$$\tilde{U} = x\, e^{it}, \tag{4.3}$$

$$\tilde{V} = -y\, e^{it}, \tag{4.4}$$

at infinity (this point will be discussed later). Using a similar method as in the previous section, the equations governing the inner unsteady flow are

$$\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial \eta^2}\right)\frac{\partial^2}{\partial \eta^2}\tilde{\psi}_0 = 0, \tag{4.5}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial \eta^2}\right)\frac{\partial^2}{\partial \eta^2}\tilde{\psi}_1 = 0, \tag{4.6}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial \eta^2}\right)\frac{\partial^2}{\partial \eta^2}\tilde{\psi}_2 = \left(2\frac{\partial^4}{\partial x^2\,\partial \eta^2}-\frac{\partial^3}{\partial t\,\partial x^2}\right)\tilde{\psi}_0-\left[\frac{\partial(\tilde{\psi}_0,\tilde{\psi}_{0\eta\eta})}{\partial(\eta,x)}\right]_u, \tag{4.7}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial \eta^2}\right)\frac{\partial^2}{\partial \eta^2}\tilde{\psi}_3 = \left(2\frac{\partial^4}{\partial x^2\,\partial \eta^2}-\frac{\partial^3}{\partial t\,\partial x^2}\right)\tilde{\psi}_1-\left[\frac{\partial(\tilde{\psi}_0,\tilde{\psi}_{1\eta\eta})}{\partial(\eta,x)}\right]_u-\left[\frac{\partial(\tilde{\psi}_1,\tilde{\psi}_{0\eta\eta})}{\partial(\eta,x)}\right]_u, \tag{4.8}$$

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial \eta^2}\right)\frac{\partial^2}{\partial \eta^2}\tilde{\psi}_4 = {} & \left(2\frac{\partial^4}{\partial x^2\,\partial \eta^2}-\frac{\partial^3}{\partial t\,\partial x^2}\right)\tilde{\psi}_2+\frac{\partial^4}{\partial x^4}\tilde{\psi}_0-\left[\frac{\partial(\overline{\psi}_0,\tilde{\psi}_{0\eta\eta})}{\partial(\eta,x)}+\frac{\partial(\tilde{\psi}_0,\overline{\psi}_{0\eta\eta})}{\partial(\eta,x)}\right] \\ & -\left[\frac{\partial(\tilde{\psi}_0,\tilde{\psi}_{2\eta\eta})}{\partial(\eta,x)}+\frac{\partial(\tilde{\psi}_2,\tilde{\psi}_{0\eta\eta})}{\partial(\eta,x)}+\frac{\partial(\tilde{\psi}_1,\tilde{\psi}_{1\eta\eta})}{\partial(\eta,x)}+\frac{\partial(\tilde{\psi}_0,\tilde{\psi}_{0xx})}{\partial(\eta,x)}\right]_u.\end{aligned} \tag{4.9}$$

For the inner steady flow the equations are:

$$\frac{\partial^4}{\partial \eta^4}\overline{\psi}_0 = \left[\frac{\partial(\tilde{\psi}_0,\tilde{\psi}_{0\eta\eta})}{\partial(\eta,x)}\right]_s, \tag{4.10}$$

$$\frac{\partial^4}{\partial \eta^4}\overline{\psi}_1 = \left[\frac{\partial(\tilde{\psi}_1,\tilde{\psi}_{0\eta\eta})}{\partial(\eta,x)}+\frac{\partial(\tilde{\psi}_0,\tilde{\psi}_{1\eta\eta})}{\partial(\eta,x)}\right]_s, \tag{4.11}$$

$$\frac{\partial^4}{\partial \eta^4}\overline{\psi}_2 = -2\frac{\partial^4}{\partial \eta^2\,\partial x^2}\overline{\psi}_0+\left[\frac{\partial(\tilde{\psi}_0,\tilde{\psi}_{2\eta\eta})}{\partial(\eta,x)}+\frac{\partial(\tilde{\psi}_2,\tilde{\psi}_{0\eta\eta})}{\partial(\eta,x)}+\frac{\partial(\tilde{\psi}_1,\tilde{\psi}_{1\eta\eta})}{\partial(\eta,x)}+\frac{\partial(\tilde{\psi}_0,\tilde{\psi}_{0xx})}{\partial(\eta,x)}\right]_s, \tag{4.12}$$

where $y = \epsilon\eta$. Using the method of inner and outer expansions, we have to solve alternatively for the unsteady flow and the steady streaming. The interactions are clearly shown in the above equations.

Without going into the details, the results are:

$$\tilde{\Psi}_0 = xy\,e^{it}, \tag{4.13}$$

$$\tilde{\psi}_0 = x\,e^{it}\left[\eta+\frac{i-1}{\sqrt{2}}(1-E)\right], \tag{4.14}$$

$$\tilde{\Psi}_1 = \frac{i-1}{\sqrt{2}}x\,e^{it}, \tag{4.15}$$

$$\tilde{\psi}_1 = \tilde{\Psi}_2 = 0, \tag{4.16}$$

$$\tilde{\psi}_2 = \frac{x\,e^{2it}}{2}\left[\frac{1+i}{2}(E^{\sqrt{2}}-1)+i\eta E\right], \tag{4.17}$$

$$\tilde{\Psi}_3 = -\frac{(1+i)\,e^{2it}}{4}x, \tag{4.18}$$

$$\tilde{\psi}_3 = \tilde{\Psi}_4 = 0, \tag{4.19}$$

$$\overline{\psi}_0 = -x\left[\tfrac{3}{4}\eta-\frac{13}{4\sqrt{2}}+\frac{1}{4\sqrt{2}}e^{-\sqrt{2}\eta}+\frac{3+2i}{\sqrt{2}}E+\frac{i}{2}\eta E\right], \tag{4.20}$$

$$\overline{\Psi}_0 = -\tfrac{3}{4}xy, \tag{4.21}$$

$$\overline{\Psi}_1 = \frac{13}{4\sqrt{2}}x, \tag{4.22}$$

$$\overline{\psi}_1 = \overline{\Psi}_2 = \overline{\psi}_2 = \overline{\Psi}_3 = \overline{\psi}_3 = \overline{\Psi}_4 = 0, \tag{4.23}$$

$$\tilde{\psi}_4 = \frac{x}{4}\left[e^{3it}\int_0^\eta g\,d\eta + e^{it}\int_0^\eta h\,d\eta\right], \tag{4.24}$$

etc., where

$$g(\eta) = \left\{\left(\frac{1}{\sqrt{2}}-\frac{3}{4}\right)E - (\sqrt{2}+2) + (1+i)\,\eta\right\}E^{\sqrt{2}} + \left\{\left(2-\frac{1}{2\sqrt{2}}\right) - \frac{3(1+i)}{2\sqrt{2}}\eta + \frac{i}{2}\eta^2\right\}E + 3\left(\frac{1}{4}+\frac{1}{2\sqrt{2}}\right)E^{\sqrt{3}}, \tag{4.25}$$

$$\begin{aligned} h(\eta) = {} & \left\{\frac{21i-13}{5} + \sqrt{2}i\eta - \frac{3+i}{20}E\right\}e^{-\sqrt{2}\eta} - \frac{(13-3i)}{4\sqrt{2}}\eta E \\ & + \left\{-(\tfrac{3}{4}+i)\,\eta^2 + \left(\frac{7i}{2}+\frac{i}{2\sqrt{2}}+\frac{13}{2}\right) + \frac{(9-5i)}{2\sqrt{2}}\eta - \tfrac{1}{2}i\eta^2\right\}E^* \\ & + \left\{(\sqrt{2}\,i-6) - (1+i)\,\eta + \left(\frac{1}{4}+\frac{i}{\sqrt{2}}\right)E^*\right\}E^{\sqrt{2}} \\ & + \left(2 - \frac{153}{20}i - \frac{7}{2\sqrt{2}}i\right)E \end{aligned} \tag{4.26}$$

and E^* is the complex conjugate of E.

The outer induced flows are governed by (3.9) and (3.11). We have prescribed for the boundary conditions at infinity that the solution be no more singular than the original oscillatory flow, and we have used the least singular solution. There is a certain degree of non-uniqueness to the outer solution because of the infinite geometry. This difficulty can be resolved by taking an initial-value problem (start oscillating from rest) and taking the time limit, or by taking a finite body and letting the radius of curvature approach infinity. Using a limiting process on the results of the oscillating cylinder obtained in the previous section, one can show that to the order considered, $O(S^{-2})$, the least singular solution *is* the correct solution.

Uniformly valid composite solutions are constructed for the oscillatory flow towards a stagnation point:

$$\begin{aligned} \tilde{\psi} = {} & xy\,e^{it} + S^{-\frac{1}{2}}\frac{i-1}{\sqrt{2}}x\,e^{it}[1-E] + S^{-\frac{3}{2}}\frac{x\,e^{2it}}{2}\left[\frac{1+i}{2}(E^{\sqrt{2}}-1) + i\eta E\right] \\ & + S^{-\frac{5}{2}}\frac{x}{4}\left[e^{3it}\int g\,d\eta + e^{it}\int h\,d\eta\right] + O(S^{-\frac{7}{2}}), \end{aligned} \tag{4.27}$$

$$\overline{\psi} = -S^{-1}\tfrac{3}{4}xy - S^{-\frac{3}{2}}x\left[-\frac{13}{4\sqrt{2}} + \frac{1}{4\sqrt{2}}e^{-\sqrt{2}\eta} + \frac{3+2i}{\sqrt{2}}E + \frac{i}{2}\eta E\right] + O(S^{-\frac{7}{2}}). \tag{4.28}$$

The steady streaming is caused by the steady component of the Reynolds stress. The two terms in (4.28) adequately describe the flow up to $O(S^{-\frac{7}{2}})$. The steady streamfunction for $S = 100$ is plotted in figure 4. We see that a layer of reverse flow always exists near the solid boundary. Also shown in the figure in dashed lines is the forcing function (Reynolds stress and pressure gradient) which drives this steady flow. The existence of a reverse flow is due to the fact that near the solid surface the Reynolds stress is negligible and the pressure gradient dominates. The thickness of the reverse flow decreases with increasing frequency of oscillation. Outside the boundary layer the forcing function decays to zero and the fluid is being dragged along by the non-zero velocity created in the boundary layer.

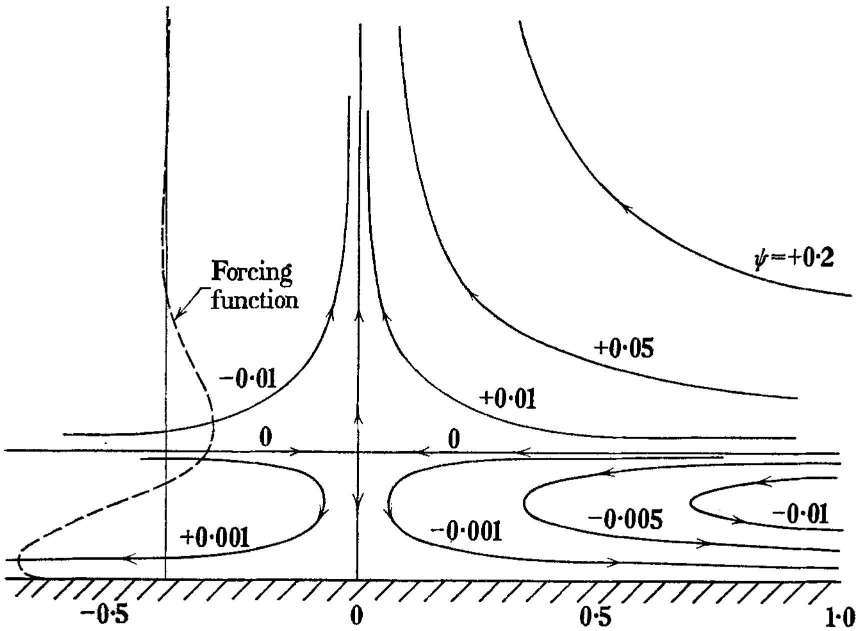

FIGURE 4. The steady streaming caused by an oscillatory flow towards a flat plate, $S = 100$.

5. Conclusions

In studying oscillatory viscous flows, the importance of separating the steady part and the unsteady part of the governing equations cannot be over-emphasized. Aside from a determination of the order of magnitude of steady streaming, this method has the advantages of showing precisely when interactions should be considered, and also providing a governing equation for the outer steady flow. The difficulties encountered in many previous investigations can partially be resolved.

By restricting ourselves to high-frequency oscillations, we are able to obtain uniformly valid analytic solutions. This is because the flow is primarily diffusive and the troublesome non-linear terms appear only as a forcing function. The procedure is further simplified by the use of the method of inner and outer expansions.

The effects of curvature are found to be very important in the case of an oscillating circular cylinder at low Reynolds number and high frequency. The interactions between the boundary layer and the outer flow field cannot be neglected. Only from the attenuation of the outer flow can one find the correct pressure drag.

The oscillating flow at a stagnation point is investigated. The solutions are valid for all Reynolds numbers in a region very near the stagnation point of the unsteady flow. We find that, as the Reynolds stress forces a steady streaming towards the stagnation point, the pressure gradient acts in the opposite direction, causing reverse flows or cells near the solid boundary.

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